COMPLEX THEOREM

Statement

(i) if $n \in Z$ (the set of integers), then $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(ii) if $n \in Q$ (the set of rational number), then $\cos(n\theta) + i\sin(n\theta)$ one of the values of $(\cos\theta + i\sin\theta)^n$.

Roots of Unity

Let $z = a + ib$ be a complex number, and let $r(\cos\theta + i\sin\theta)$ be the polar form of z.

Then by De Moivre's theorem $r^{1/n}\left\{\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right\}$ is one of the values of $z^{1/n}$.

Cube Roots of unity

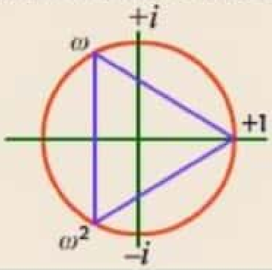
$z = (1)^{1/3}$

Roots : $1, \omega, \omega^2$, where $\omega = e^{i\frac{2\pi}{3}}$

Properties of Cube Roots of Unity

- $1 + \omega^r + \omega^{2r} = 0 \qquad r \neq 3n$
- $\omega = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

  $\omega^2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
- The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle.



n^{th} Roots of unity

$z = (1)^{1/n}$

Roots : $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

$\alpha_r = e^{i\frac{2\pi r}{n}} = \cos\frac{2\pi r}{n} + i\sin\frac{2\pi r}{n}$

Properties of n^{th} Roots of Unity

- They are in G.P. with common ratio $e^{i\frac{2\pi}{n}}$
- $1^p + \alpha_1^p + \alpha_2^p + \ldots + \alpha_{n-1}^p = 0$ if $p \neq kn$
- $1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots + (\alpha_{n-1})^p = n$ if $p = kn$
- $(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_{n-1}) = n$
- $(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd
- $(1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots \alpha_{n-1}) = (-1)^{n-1}$
- $(\omega-\alpha_1)(\omega-\alpha_2)\ldots(\omega-\alpha_{n-1}) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+\omega & \text{if } n = 3k+2 \end{cases}$

Point to Remember

Centroid, Incentre, Orthocentre & Circumcentre of a triangle on a complex plane

(a) Centroid 'G' $= \frac{z_1 + z_2 + z_3}{3}$

(b) Incentre 'I' $= \frac{a z_1 + b z_2 + c z_3}{a + b + c}$

(c) Orthocentre 'Z_H' $= \frac{Z_1\tan A + Z_2\tan B + Z_3\tan C}{\tan A + \tan B + \tan C}$

(D) Circumcentre 'Z_S' $= \frac{Z_1(\sin 2A) + Z_2(\sin 2B) + Z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$